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Reduced-order observer-based control design for nonlinear stochastic systems☆

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Abstract

In this paper, we investigate the stabilization control design problem of nonlinear stochastic SISO systems in strict-feedback form. By introducing a novel reduced-order observer, an output-feedback-based control is constructively designed, which renders the closed-loop system asymptotically stable in the large when the nonlinearities and stochastic disturbance equal zero at the equilibrium point of the open-loop system, and bounded in probability, otherwise. Besides, the obtained controller preserves the equilibrium point of the open-loop nonlinear system.

Keywords: Nonlinear system; Stochastic system; Reduced-order observer; Integrator backstepping; Stabilization control

1. Introduction

Global stabilization control design for stochastic nonlinear systems has been being a research topic under intensive investigation recently (e.g. [1–3,8–10]), which is based on recursive applications of cascade designs, such as the well-known integrator backstepping method. Khas'minski and Kushner, etc. presented the basic stability theory of the stochastic control systems in their classical books [6,7], and introduced two important stability notions, *bounded in probability* and *asymptotically stable in the large*, which have now been widely applied. It is well known that how to deal with the quadratic variation terms is the key to stochastic control design. Methods in existence is realized via restraining the stochastic disturbance by increasing the power of the state variables in control laws (e.g. [1,3]) or enlarging the power of the feedback capacity (e.g. [8,9]). By using quartic Lyapunov function, asymptotical stabilization control in the large was presented in [1–3] under the assumption that the nonlinearities and disturbance equal zero at the equilibrium point of the open-loop system. In [8,9], the authors investigated the optimal control design problem under risk-sensitive cost function, and pointed out that if the control goal is to stabilize the closed-loop system, then it is not necessary to require

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that the nonlinearities and disturbance equal zero at the equilibrium point of the open-loop system, although it seems unavoidable for asymptotical stabilization control.

The controls in [3,9] were based on complete state feedback, and the controls in [1,8,10] were based on output feedback and full-order state observers. From [1,8,10] it can be seen that when the stochastic noise equals zero, the state estimation error depends only on the initial condition instead of the output or state process, and converges asymptotically to zero. Recently, in [5], Jiang gave a *reduced-order* observer with a special structure for deterministic systems. Compared with [1,8,10], an extra nonlinear term depending on the output process appears in the state estimation error equation. Generally speaking, this nonlinear term is not zero, and may affect the asymptotical convergence of the state estimation error, even when there is no stochastic disturbance.

In this paper, we study the output-feedback stabilization control design problem. By introducing a novel reduced-order observer, the above-mentioned extra nonlinear term appeared in the state estimation error equation (e.g. [5]) is removed, and at the same time, all the advantages on the asymptotical convergence of the state estimation error of full-order state observers (e.g. [1,8,10]) are preserved.

The paper is organized as follows. In Section 2, some notations and preliminary results are introduced. In Section 3, the problem to be studied is formulated. In Section 4, a novel reduced-order observer is presented first, and then a control is constructively designed to ensure the closed-loop system bounded in probability or asymptotically stable in the large, depending on the conditions on the nonlinearities and stochastic disturbance, respectively. In Section 5, an example is given to illustrate the design method proposed in this paper. In Section 6, some conclusion remarks are given.

2. Notations and preliminary results

In the sequel, we will use the following notations. For a given vector or matrix X, X^τ denotes its transpose; $\|X\|$ denotes the Euclidean norm for vectors and the corresponding induced norm for matrices; $\mathrm{tr}(X)$ denotes its trace when X is square. For a given vector $x=(x_1,\ldots,x_n)^\tau$, $x_{[i]}$ denotes $(x_1,\ldots,x_i)^\tau$; $x_{[i,j]}$ denotes $(x_i,\ldots,x_j)^\tau$; $\hat{x}$ denotes is its estimate associated with a observer, and $\tilde{x}$ denotes the estimation error, i.e. $\tilde{x}=x-\hat{x}$. For a given scalar number x, $|x|$ denotes its absolute value. I_i denotes the identity matrix with i-dimension. $\mathscr{C}^\infty$ denotes the set of all infinitely differentiable functions. For simplicity of expression, we sometimes omit the arguments of functions when no confusion is caused.

For systems of the form

$$\mathrm{d}x = f(x)\,\mathrm{d}t + h(x)\,\mathrm{d}w, \tag{1}$$

where $f(\cdot)$ and $h(\cdot)$ are locally Lipschitz in x, and w is vector-valued Brownian motion defined below, we define a differential operator $\mathscr{L}$ for twice continuously differentiable function $V(x)$ as follows:

$$\mathscr{L}V(x) = \frac{\partial V(x)}{\partial x} f(x) + \tfrac{1}{2}\,\mathrm{tr}\left\{\frac{\partial^2 V(x)}{\partial x^2} h(x) h^\tau(x)\right\}.$$

Recall two stability notions for nonlinear stochastic system (1).

Definition 1 (Khas'minski [6]). Consider system (1) with $f(0)=0$ and $h(0)=0$. The solution $x(t)=0$ is said to be asymptotically stable in the large if for any $\varepsilon>0$,

$$\lim_{x(0)\to 0} P\left\{\sup_{t\geqslant 0}\|x(t)\| \geqslant \varepsilon\right\} = 0$$

and for any initial condition $x(0)$,

$$P\left\{\lim_{t\to\infty} x(t) = 0\right\} = 1.$$

Definition 2 (Khas'minski [6]). The solution process $\{x(t), t \geqslant 0\}$ of stochastic differential system (1) is said to be bounded in probability, if

$$\lim_{c\to\infty} \sup_{0\leqslant t<\infty} P\{\|x(t)\| > c\} = 0.$$

Corresponding to these concepts, we have the following basic theorem, which will play an important role in our control design below.

Theorem 1. *Consider the stochastic nonlinear system* (1). *If there exists a positive definite, radially unbounded, twice continuously differentiable Lyapunov* $V : \mathbb{R}^n \to \mathbb{R}$, *and constants* $c_1 > 0,\ c_2 \geqslant 0$ *such that*

$$\mathscr{L}V(x) \leqslant -c_1 V(x) + c_2, \tag{2}$$

then

1. *the system has a unique solution almost surely*;
2. *the system is bounded in probability*;
3. *in addition, if* $f(0) = 0, h(0) = 0$ *and*

$$\mathscr{L}V(x) \leqslant -c_1 V(x), \tag{3}$$

the system is asymptotically stable in the large.

Proof. By Theorem 4.1 of Chapter 3 of [6], Theorem 4.4 of Chapter 5 of [6], Theorem 2 of Chapter 3 of [4] and Section 13 of [4], we can show the theorem in a similar way proving Theorem 2.5 of [9]. □

3. Problem formulation

Consider the following nonlinear stochastic system:

$$\begin{aligned}
\mathrm{d}x_1 &= x_2\,\mathrm{d}t + f_1(x_1)\,\mathrm{d}t + \varphi_1(x_1)\,\mathrm{d}w,\\
\mathrm{d}x_2 &= x_3\,\mathrm{d}t + f_2(x_{[2]})\,\mathrm{d}t + \varphi_2(x_{[2]})\,\mathrm{d}w,\\
&\vdots\\
\mathrm{d}x_r &= x_{r+1}\,\mathrm{d}t + f_r(y)\,\mathrm{d}t + \varphi_r(y)\,\mathrm{d}w,\\
\mathrm{d}x_{r+1} &= x_{r+2}\,\mathrm{d}t + f_{r+1}(y)\,\mathrm{d}t + \varphi_{r+1}(y)\,\mathrm{d}w,\\
&\vdots\\
\mathrm{d}x_{n-1} &= x_n\,\mathrm{d}t + f_{n-1}(y)\,\mathrm{d}t + \varphi_{n-1}(y)\,\mathrm{d}w,\\
\mathrm{d}x_n &= u\,\mathrm{d}t + f_n(y)\,\mathrm{d}t + \varphi_n(y)\,\mathrm{d}w,\\
y &= x_{[r]},
\end{aligned} \tag{4}$$

where y is the output of the system, which is available for feedback control design.

In [5], Jiang studied the case $r = 1$. Here we will discuss the general case $r = 1, \ldots,$ or n.

Suppose that system (4) satisfies the following assumptions:

(A1) $w \in \mathbb{R}^s$ is an independent vector-valued standard Brownian motion defined on probability space $(\Omega, \mathscr{F}, \mathscr{P})$, with Ω a sample space, $\mathscr{F}$ a $\sigma-$algebra, $\mathscr{P}$ a probability measure.
(A2) The nonlinear functions $f_i(\cdot) \in \mathscr{C}^\infty$ and $\varphi_i(\cdot) \in \mathscr{C}^\infty$ $(i = 1, \ldots, n)$.
(A3) The nonlinear function $f_i(\cdot)$ $(i = 1, \ldots, n)$ equals zero at the origin, i.e. $f_i(0) = 0$ $(i = 1, \ldots, n)$.

System (4) can be rewritten into the following compact form:

$$\begin{cases} \mathrm{d}y = A_r y\,\mathrm{d}t + B_r x_{r+1}\,\mathrm{d}t + F_r(y)\,\mathrm{d}t + H_r(y)\,\mathrm{d}w, \\ \mathrm{d}x_{[r+1,\,n]} = A_{n-r} x_{[r+1,\,n]}\,\mathrm{d}t + B_{n-r} u\,\mathrm{d}t + F_{r+1,n}(y)\,\mathrm{d}t + H_{r+1,n}(y)\,\mathrm{d}w, \end{cases} \tag{5}$$

where and whereafter, for any integer $i \in \{1, \ldots, n\}$,

$$A_i = \begin{bmatrix} 0 & I_{i-1} \\ \vdots & \\ 0 & 0\cdots 0 \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} i-1 \\ \\ \end{matrix}, \quad C_i = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \\ \left.\vphantom{\begin{matrix}0\\ \vdots\\ 0\end{matrix}}\right\} i-1 \end{matrix}$$

and

$$F_i = \begin{bmatrix} f_1 \\ \vdots \\ f_i \end{bmatrix}, \quad H_i = \begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_i \end{bmatrix}, \quad F_{i,n} = \begin{bmatrix} f_i \\ \vdots \\ f_n \end{bmatrix}, \quad H_{i,n} = \begin{bmatrix} \varphi_i \\ \vdots \\ \varphi_n \end{bmatrix}.$$

To inspire our observer design mechanism, we first give a conventional observer following [1,10] and a reduced-order observer following [5], respectively.

As stated at the beginning of the last section, we will use $\hat{x}_{[r,n]}$ to denote the estimate of $x_{[r,n]}$. Following the lines of [1,10], we can design an observer as

$$\dot{\hat{x}}_{[r,n]} = A_{n-r+1}\hat{x}_{[r,n]} + K_{r,n}(x_r - \hat{x}_r) + B_{n-r+1}u + F_{r,n}(y), \tag{6}$$

where $K_{r,n} = (k_r, \ldots, k_n)^\tau$ are design parameters such that the polynomial $s^{n-r+1} + k_r s^{n-r} + \cdots + k_{n-1}s + k_n$ is *Hurwitz*.

Let $\tilde{x}_{[r,n]} = x_{[r,n]} - \hat{x}_{[r,n]}$ be the estimation error. Then

$$\mathrm{d}\tilde{x}_{[r,n]} = S_r \tilde{x}_{[r,n]}\,\mathrm{d}t + H_{r,n}(y)\mathrm{d}w, \tag{7}$$

where S_r is strictly stable, which, for any integer $i \in \{1, \ldots, n\}$, is defined as follows:

$$S_i = \begin{bmatrix} -k_i & I_{n-i} \\ \vdots & \\ -k_n & 0\cdots 0 \end{bmatrix}.$$

Therefore, the overall system with observer (6) in the loop is

$$\begin{aligned}
\mathrm{d}\tilde{x}_{[r,n]} &= S_r\tilde{x}_{[r,n]}\,\mathrm{d}t + H_{r,n}(y)\,\mathrm{d}w,\\
\mathrm{d}x_{[r-1]} &= A_{r-1}x_{[1,r-1]}\,\mathrm{d}t + B_{r-1}x_r\,\mathrm{d}t + F_{r-1}(x_{[r-1]})\,\mathrm{d}t + H_{r-1}(x_{[r-1]})\,\mathrm{d}w,\\
\mathrm{d}x_r &= (\tilde{x}_{r+1} + \hat{x}_{r+1})\,\mathrm{d}t + f_r(y)\,\mathrm{d}t + \varphi_r(y)\,\mathrm{d}w,\\
\mathrm{d}\hat{x}_{[r+1,n]} &= A_{n-r}\hat{x}_{[r+1,n]}\,\mathrm{d}t + B_{n-r}u\,\mathrm{d}t + K_{r+1,n}\tilde{x}_r\,\mathrm{d}t + F_{r+1,n}(y)\,\mathrm{d}t.
\end{aligned}$$

Although r states $(x_1,\ldots,x_r)$ are directly available for feedback control design and only $n-r$ states $(x_{r+1},\ldots,x_n)$ are needed to be estimated, observer (6) gives $(n-r+1)$ state estimates, including one for the available state x_r, which is unnecessary and results in a redundant dimension of the observer. To eliminate estimating x_r and get a reduced-order observer with $(n-r)$-dimension, which is actually of minimal-order in linear system case [11], one natural way is to generalize the reduced-order observer developed in [5] from deterministic systems to stochastic systems as follows:

$$\begin{aligned}
\dot{\xi}_{r+1} &= \xi_{r+2} + k_{r+2}x_r - k_{r+1}(\xi_{r+1} + k_{r+1}x_r),\\
\dot{\xi}_i &= \xi_{i+1} + k_{i+1}x_r - k_i(\xi_{r+1} + k_{r+1}x_r), \quad i = r+2,\ldots,n-1,\\
\dot{\xi}_n &= u - k_n(\xi_{r+1} + k_{r+1}x_r), \qquad (8)
\end{aligned}$$

where k_i $(r+1 \leqslant i \leqslant n)$ are chosen such that matrix S_{r+1} is strictly stable.

Let $\hat{x}_{[r+1,n]} = (\xi_{r+1} + k_{r+1}x_r,\ldots,\xi_n + k_nx_r)^\tau$ be the estimate of $x_{[r+1,n]}$. Then, the estimation error $\tilde{x}_{[r+1,n]} = x_{[r+1,n]} - \hat{x}_{[r+1,n]}$ satisfies

$$\mathrm{d}\tilde{x}_{[r+1,n]} = S_{r+1}\tilde{x}_{[r+1,n]}\,\mathrm{d}t + \bar{f}(y)\,\mathrm{d}t + \bar{\varphi}(y)\,\mathrm{d}w, \qquad (9)$$

where

$$\begin{aligned}
\bar{f}(y) &= (f_{r+1}(y) - k_{r+1}f_r(y),\ldots,f_n(y) - k_nf_r(y))^\tau,\\
\bar{\varphi}(y) &= (\varphi_{r+1}(y) - k_{r+1}\varphi_r(y),\ldots,\varphi_n(y) - k_n\varphi_r(y))^\tau.
\end{aligned}$$

Remark 1. Compared (9) with (7), it is easy to see that an extra nonlinear term $\bar{f}(y)\,\mathrm{d}t$ arises. Due to this unexpected term, the estimation error $\tilde{x}_{[r+1,n]}$ may not be convergent to zero, even when $\bar{\varphi}(y)\,\mathrm{d}w \equiv 0$. In other words, when $H_{r,n}(y)\mathrm{d}w \equiv 0$ and $\bar{\varphi}(y)\,\mathrm{d}w \equiv 0$, $\tilde{x}_{[r,n]}$ given by (7) is convergent to zero, but $\tilde{x}_{[r+1,n]}$ given by (9) may not be.

4. Output-feedback control design

In this section, we will introduce a new reduced-order observer first, and then present a constructive procedure for stabilization control design. The observer introduced is of minimum-order, in which no component of $x_{[r]}$ is estimated. Besides, the observer preserves a nice structure similar to (6).

4.1. Observer design

Denote

$$q(t) \triangleq \mathrm{d}y - A_ry\,\mathrm{d}t - F_r(y)\,\mathrm{d}t,$$

which will play an important role in our observer design below. Then, by (5),

$$q(t) = B_r x_{r+1}\,\mathrm{d}t + H_r(y)\,\mathrm{d}w.$$

When $q(t)$ is available, the reduced-order observer can be designed as

$$\mathrm{d}\hat{x}_{[r+1,n]} = D\hat{x}_{[r+1,n]}\,\mathrm{d}t + B_{n-r}u\,\mathrm{d}t + F_{r+1,n}(y)\,\mathrm{d}t + Gq(t), \tag{10}$$

where $D = A_{n-r} - GB_rC_{n-r}^{\tau}$ with $G = [g_{ij}] \in \mathbb{R}^{(n-r)\times r}$ a parameter matrix to be specified so that D is strictly stable. Unfortunately, due to the existence of the unmeasurable state x_{r+1} and stochastic disturbance w, Eq. (10) is unfeasible for feedback design. To overcome this difficulty, we introduce a new state vector

$$\xi = \hat{x}_{[r+1,n]} - Gy, \tag{11}$$

which together with (10) gives

$$\dot{\xi} = D\xi + B_{n-r}u + F_{r+1,n}(y) - GF_r(y) + (DG - GA_r)y. \tag{12}$$

Obviously, ξ is feasible for control design.

By (11) we have $\hat{x}_{[r+1,n]} = \xi + Gy$ and

$$\tilde{x}_{[r+1,n]} = x_{[r+1,n]} - \hat{x}_{[r+1,n]} = x_{[r+1,n]} - \xi - Gy, \tag{13}$$

which implies

$$\begin{aligned}\mathrm{d}\tilde{x}_{[r+1,n]} &= A_{n-r}x_{[r+1,n]}\,\mathrm{d}t + B_{n-r}u\,\mathrm{d}t + F_{r+1,n}(y)\,\mathrm{d}t + H_{r+1,n}(y)\,\mathrm{d}w\\ &\qquad - \{D\xi + B_{n-r}u + F_{r+1,n}(y) - GF_r(y) + (DG - GA_r)y\}\,\mathrm{d}t\\ &\qquad - G[A_r y\,\mathrm{d}t + B_rC_{n-r}^{\tau}x_{[r+1,n]}\,\mathrm{d}t + F_r(y)\mathrm{d}t + H_r(y)\,\mathrm{d}w]\\ &= D\tilde{x}_{[r+1,n]}\,\mathrm{d}t + \bar{H}(y)\,\mathrm{d}w,\end{aligned} \tag{14}$$

where $\bar{H}(y) = H_{r+1,n}(y) - GH_r(y)$.

If the last column of matrix G is equal to $(k_{r+1},\ldots,k_n)^{\tau}$ and the polynomial $s^{n-r}+k_{r+1}s^{n-r-1}+\cdots+k_{n-1}s+k_n$ is *Hurwitz*, then the $(n-r)$-dimensional matrix

$$D = \begin{bmatrix} -k_{r+1} & \\ & I_{n-r-1} \\ \vdots & \\ & \\ -k_n & 0\cdots 0 \end{bmatrix}$$

is strictly stable. And so, there exists a positive definite matrix P satisfying

$$D^{\tau}P + PD = -I_{n-r}.$$

It follows from (14) that if the stochastic disturbance equals zero, i.e. $\bar{H}(y)\,\mathrm{d}w \equiv 0$, then the estimation error $\tilde{x}_{[r+1,n]}$ converges to zero. So, by (13), $(\xi + Gy)$ can be used to estimate the unmeasurable states $(x_{r+1},\ldots,x_n)^{\tau}$.

The overall system with the observer in the loop is

$$\begin{aligned}&\mathrm{d}\tilde{x}_{[r+1,n]} = D\tilde{x}_{[r+1,n]}\,\mathrm{d}t + \bar{H}(y)\,\mathrm{d}w,\\ &\quad \mathrm{d}y = A_r y\,\mathrm{d}t + \tilde{x}_{r+1}\,\mathrm{d}t + \xi_1\,\mathrm{d}t + B_rC_{n-r}^{\tau}Gy\,\mathrm{d}t + F_r(y)\,\mathrm{d}t + H_r(y)\,\mathrm{d}w,\\ &\quad \mathrm{d}\xi = D\xi\,\mathrm{d}t + B_{n-r}u\,\mathrm{d}t + F_{r+1,n}(y)\,\mathrm{d}t - GF_r(y)\,\mathrm{d}t + (DG - GA_r)y\,\mathrm{d}t,\end{aligned} \tag{15}$$

which is suitable for backstepping design.

4.2. Feedback control design

We are now in a position to construct a control $u(y,\xi)\in\mathscr{C}^\infty$ for the overall system (15) to ensure the closed-loop system bounded in probability and asymptotically stable in the large when the nonlinearities and disturbance equal zero at the equilibrium point of the open-loop system.

Introduce a diffeomorphic state transform as follows:

$$z_1 = y,\ z_i = x_i - \alpha_{i-1}(x_{[i-1]}),\quad 2\leqslant i\leqslant r,$$
$$z_i = \xi_{i-r} - \alpha_{i-1}(y,\xi_{[i-r-1]}),\quad r+1\leqslant i\leqslant n,\ z_{n+1}=0, \tag{16}$$

where α_i $(1\leqslant i\leqslant n-1)$ are $\mathscr{C}^\infty$ functions and will be designed as virtual controls, $\alpha_n = u(y,\ \xi)$ is the real control law to be specified later.

Under the new variable vector z, system (15) becomes

$$\mathrm{d}\tilde{x}_{[r+1,n]} = D\tilde{x}_{[r+1,n]}\,\mathrm{d}t + \bar{H}(y)\,\mathrm{d}w,$$
$$\mathrm{d}z_i = (z_{i+1}+\alpha_i)\,\mathrm{d}t + \Omega_i(x_{[i]})\,\mathrm{d}t + \Phi_i(x_{[i]})\,\mathrm{d}w,\quad i=1,\ldots,r-1,$$
$$\mathrm{d}z_r = (z_{r+1}+\alpha_r)\,\mathrm{d}t + \tilde{x}_{r+1}\,\mathrm{d}t + \Omega_r(y)\,\mathrm{d}t + \Phi_r(y)\,\mathrm{d}w,$$
$$\mathrm{d}z_{r+i} = (z_{r+i+1}+\alpha_{r+i})\,\mathrm{d}t - \frac{\partial\alpha_{r+i-1}}{\partial x_r}\tilde{x}_{r+1}\,\mathrm{d}t + \Omega_{r+i}(y,\xi_{[i]})\,\mathrm{d}t + \Phi_{r+i}(y,\xi_{[i-1]})\,\mathrm{d}w,\quad i=1,\ldots,n-r-1,$$
$$\mathrm{d}z_n = u\,\mathrm{d}t - \frac{\partial\alpha_{n-1}}{\partial x_r}\tilde{x}_{r+1}\,\mathrm{d}t + \Omega_n(y,\xi_{[n-r]})\,\mathrm{d}t + \Phi_n(y,\xi_{[n-r-1]})\,\mathrm{d}w, \tag{17}$$

where

$$\Phi_1 = \varphi_1(x_1),\qquad \Phi_i = \varphi_i(x_{[i]}) - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}\,\varphi_j(x_{[j]}),\quad i=2,\ldots,r,$$

$$\Phi_{r+i} = \varphi_{r+i}(y) - \sum_{j=1}^{r}\frac{\partial\alpha_{r+i-1}}{\partial x_j}\,\varphi_j(x_{[j]}),\quad i=1,\ldots,n-r,$$

$$\Omega_i = f_i(x_{[i]}) - \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}[x_{j+1}+f_j(x_{[j]})] - \frac{1}{2}\sum_{j,k\in\{1,\ldots,i-1\}}\frac{\partial^2\alpha_{i-1}}{\partial x_j\partial x_k}\,\varphi_j\varphi_k^\tau,\quad i=1,\ldots,r-1,$$

$$\Omega_r = f_r(y) + \sum_{j=1}^{r}g_{1j}x_j - \sum_{j=1}^{r-1}\frac{\partial\alpha_{r-1}}{\partial x_j}[x_{j+1}+f_j] - \frac{1}{2}\sum_{j,k\in\{1,\ldots,r-1\}}\frac{\partial^2\alpha_{i-1}}{\partial x_j\partial x_k}\varphi_j\varphi_k^\tau,$$

$$\begin{aligned}\Omega_{r+i} = {}& f_{r+i}(y) - k_{r+i}\xi_1 - \sum_{j=1}^{r}g_{ij}f_j + [DG-GA_r]_i y - \sum_{j=1}^{r-1}\frac{\partial\alpha_{r+i-1}}{\partial x_j}x_{j+1}\\ &-\sum_{j=1}^{r}\frac{\partial\alpha_{r+i-1}}{\partial x_j}f_j(x_{[j]}) - \frac{\partial\alpha_{r+i-1}}{\partial x_r}\left(\xi_1+\sum_{i=1}^{r}g_{1i}x_i\right) - \sum_{j=1}^{i-1}\frac{\partial\alpha_{r+i-1}}{\partial\xi_j}\left(-k_{r+j}\xi_1+\xi_{j+1}+f_{r+j}\right.\\ &\left.+[DG-GA_r]_j y + \sum_{k=1}^{r}g_{jk}f_k\right) - \frac{1}{2}\sum_{j,k\in\{1,\ldots,r\}}\frac{\partial^2\alpha_{i-1}}{\partial x_j\partial x_k}\varphi_j\varphi_k^\tau,\quad i=1,\ldots,n-r.\end{aligned}$$

Here $[DG-GA_r]_i$ denotes the ith row of $DG-GA_r$.

Choose Lyapunov function $V(\tilde{x}_{[r+1,n]}, z)$ as in [8,9]:

$$V = \tilde{x}_{[r+1,n]}^{\tau} P \tilde{x}_{[r+1,n]} + \sum_{i=1}^{n} \Xi_i(z_{[i-1]}) z_i^2, \tag{18}$$

where $\Xi_i(z_{[i-1]}) > 0$ $(i = 1, \ldots, n)$ are $\mathscr{C}^{\infty}$ functions to be specified below.

By Itô formula we have

$$\begin{aligned}\mathscr{L}V = & -\|\tilde{x}_{[r+1,n]}\|^2 + \mathrm{tr}\{\bar{H}^{\tau}(y) P \bar{H}(y)\} + 2\sum_{i=1}^{n} \Xi_i (z_{i+1} + \alpha_i + \Omega_i) z_i + \sum_{i=r}^{n} M_i \tilde{x}_{r+1} z_i \\ & + \sum_{i=2}^{n} \sum_{j=1}^{i-1} \frac{\partial \Xi_i}{\partial z_j} (z_{j+1} + \alpha_j + \Omega_j) z_i^2 + \frac{1}{2} \sum_{i=1}^{n} \mathrm{tr}\left\{ \frac{\partial^2 (\Xi_i z_i^2)}{\partial z_{[i]}^2} [\Phi_1^{\tau}, \ldots, \Phi_i^{\tau}]^{\tau} [\Phi_1^{\tau}, \ldots, \Phi_i^{\tau}] \right\},\end{aligned} \tag{19}$$

where

$$M_r = 2\Xi_r, \quad M_{r+1} = \frac{\partial \Xi_{r+1}}{\partial z_r} z_{r+1} - 2\Xi_{r+1} \frac{\partial \alpha_r}{\partial x_r},$$

$$M_i = \frac{\partial \Xi_i}{\partial z_r} z_i - \sum_{j=r+1}^{i-1} \frac{\partial \Xi_i}{\partial z_j} \frac{\partial \alpha_{j-1}}{\partial x_r} z_i - 2\Xi_i \frac{\partial \alpha_{i-1}}{\partial x_r}, \quad i = r+2, \ldots, n.$$

Let $X = (y^{\tau},\ \xi^{\tau})^{\tau}$. Then by diffeomorphism (16) there exist $\mathscr{C}^{\infty}$ functions $\vartheta_i(\cdot)$ $(i = 1, \ldots, n)$ such that

$$z_{[i]} = \vartheta_i(X_{[i]}), \quad i = 1, \ldots, n.$$

Therefore, noticing that $\varphi_i(\cdot) \in \mathscr{C}^{\infty}$, $\alpha_i(\cdot) \in \mathscr{C}^{\infty}$ and $\vartheta_i(\cdot) \in \mathscr{C}^{\infty}$ $(i=1, \ldots, r)$ we see that Φ_i can be decomposed into the following forms:

$$\bar{H}(y) = \bar{\bar{H}}(z_{[r]}) = \bar{\bar{H}}(0) + \sum_{i=1}^{r} \bar{\bar{H}}_i(z_{[i]}) z_i,$$

$$\Phi_i(X_{[i]}) = \bar{\Phi}_i(z_{[i]}) = \bar{\Phi}_i(0) \quad + \sum_{j=1}^{i} \bar{\Phi}_{ij}(z_{[j]}) z_j, \quad i = 1, \ldots, n,$$

where $\bar{\bar{H}}(0) = \bar{H}(0) = H_{r+1,n}(0) - GH_r(0)$ can be used in the control design.

Then we have

$$\begin{aligned}\mathrm{tr}\{\bar{H}(y) P \bar{H}^{\tau}(y)\} &= \mathrm{tr}\left\{ \left[\bar{\bar{H}}(0) + \sum_{i=1}^{r} \bar{\bar{H}}_i(z_{[i]}) z_i \right] P \left[\bar{\bar{H}}(0) + \sum_{i=1}^{r} \bar{\bar{H}}_i(z_{[i]}) z_i \right]^{\tau} \right\} \\ &\leqslant \mathrm{tr}[\bar{\bar{H}}(0) P \bar{\bar{H}}^{\tau}(0)] + \sum_{i=1}^{r} \mathrm{tr}\{[2\bar{\bar{H}}(0) + r\bar{\bar{H}}_i(z_{[i]}) z_i] P \bar{\bar{H}}_i^{\tau}(z_{[i]})\} z_i\end{aligned} \tag{20}$$

and

$$\frac{1}{2} \sum_{i=1}^{n} \mathrm{tr}\left\{ \frac{\partial^2 (\Xi_i z_i^2)}{\partial z_{[i]}^2} [\Phi_1^{\tau}, \ldots, \Phi_i^{\tau}]^{\tau} [\Phi_1^{\tau}, \ldots, \Phi_i^{\tau}] \right\} = \frac{1}{2} \sum_{i=1}^{n} \mathrm{tr}\left[\begin{bmatrix} \dfrac{\partial^2 \Xi_i}{\partial z_{[i-1]}^2} z_i^2 & 2\dfrac{\partial \Xi_i}{\partial z_{[i-1]}} z_i \\ 2\left(\dfrac{\partial \Xi_i}{\partial z_{[i-1]}}\right)^{\tau} z_i & 2\Xi_i \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \vdots \\ \Phi_i \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \vdots \\ \Phi_i \end{bmatrix}^{\tau} \right]$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{i=1}^{n} \mathrm{tr}\left[\begin{bmatrix} \dfrac{\partial^2 \Xi_i}{\partial z_{[i-1]}^2} z_i & 2\dfrac{\partial \Xi_i}{\partial z_{[i-1]}} \\ 2\left(\dfrac{\partial \Xi_i}{\partial z_{[i-1]}}\right)^{\tau} & 0 \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \vdots \\ \Phi_i \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \vdots \\ \Phi_i \end{bmatrix}^{\tau}\right] z_i \\
&\quad + \sum_{i=1}^{n} \Xi_i \left[\bar{\Phi}_i(0) + \sum_{j=1}^{i-1} \bar{\Phi}_{ij}(z_{[j]}) z_j + \bar{\Phi}_{ii}(z_{[i]}) z_i\right] \left[\bar{\Phi}_i(0) + \sum_{j=1}^{i-1} \bar{\Phi}_{ij}(z_{[j]}) z_j + \bar{\Phi}_{ii}(z_{[i]}) z_i\right]^{\tau} \\
&\leqslant \frac{1}{2}\sum_{i=1}^{n} \mathrm{tr}\left[\begin{bmatrix} \dfrac{\partial^2 \Xi_i}{\partial z_{[i-1]}^2} z_i & 2\dfrac{\partial \Xi_i}{\partial z_{[i-1]}} \\ 2\left(\dfrac{\partial \Xi_i}{\partial z_{[i-1]}}\right)^{\tau} & 0 \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \vdots \\ \Phi_i \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \vdots \\ \Phi_i \end{bmatrix}^{\tau}\right] z_i + 3\sum_{i=1}^{n} \Xi_i \|\bar{\Phi}_{ii}(z_{[i]})\|^2 z_i^2 \\
&\quad + 3\sum_{i=1}^{n} \Xi_i \|\bar{\Phi}_i(0)\|^2 + 3\sum_{i=1}^{n} i\Xi_i \sum_{j=1}^{i-1} \Xi_j^{-1} \|\bar{\Phi}_{ij}\|^2 \Xi_j z_j^2.
\end{aligned} \tag{21}
$$

By substituting (20) and (21) into (19) we have

$$
\begin{aligned}
\mathscr{L}V &\leqslant -\|\tilde{x}_{[r+1,n]}\|^2 + \mathrm{tr}\left\{\bar{\bar{H}}(0) P \bar{\bar{H}}^{\tau}(0)\right\} + \sum_{i=1}^{r} \mathrm{tr}\left\{\left[2\bar{\bar{H}}(0) + r\bar{\bar{H}}_i(z_{[i]}) z_i\right] P \bar{\bar{H}}_i^{\tau}(z_{[i]})\right\} z_i \\
&\quad + 2\sum_{i=1}^{n} \Xi_i (z_{i+1} + \alpha_i + \Omega_i) z_i + \sum_{i=2}^{n}\sum_{j=1}^{i-1} \frac{\partial \Xi_i}{\partial z_j}(z_{j+1} + \alpha_j + \Omega_j) z_i^2 + \frac{1}{2\varepsilon_1}\sum_{i=r}^{n} M_i^2 z_i^2 \\
&\quad + \frac{(n-r+1)\varepsilon_1}{2}(\tilde{x}_{r+1})^2 + \frac{1}{2}\sum_{i=1}^{n} \mathrm{tr}\left[\begin{bmatrix} \dfrac{\partial^2 \Xi_i}{\partial z_{[i-1]}^2} z_i & 2\dfrac{\partial \Xi_i}{\partial z_{[i-1]}} \\ 2\left(\dfrac{\partial \Xi_i}{\partial z_{[i-1]}}\right)^{\tau} & 0 \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \vdots \\ \Phi_i \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \vdots \\ \Phi_i \end{bmatrix}^{\tau}\right] z_i \\
&\quad + 3\sum_{i=1}^{n} \Xi_i \|\bar{\Phi}_{ii}(z_{[i]})\|^2 z_i^2 + 3\sum_{i=1}^{n} \Xi_i \|\bar{\Phi}_i(0)\|^2 + 3\sum_{i=1}^{n} i\Xi_i \sum_{j=1}^{i-1} \Xi_j^{-1} \|\bar{\Phi}_{ij}\|^2 \Xi_j z_j^2,
\end{aligned} \tag{22}
$$

where we have used

$$
M_i \tilde{x}_{r+1} z_i \leqslant \frac{1}{2\varepsilon_1} M_i^2 z_i^2 + \frac{\varepsilon_1}{2}(\tilde{x}_{r+1})^2, \quad i = r, \ \ldots, n \ \text{and} \ \forall \varepsilon_1 > 0.
$$

Choose the weighted functions as

$$
\Xi_i = \frac{\kappa_i}{1 + \|\bar{\Phi}_i(0)\|^2 + \sum_{j=1}^{i-1} \Xi_j^{-1} \|\bar{\Phi}_{ij}\|^2}, \quad 1 \leqslant i \leqslant n,
$$

where κ_i $(i=1,\ldots,n)$ are positive constants to be specified. Then from (22) it follows that

$$\mathscr{L}V \leqslant -\bar{c}_1\|\tilde{x}_{[r+1,n]}\|^2 - \sum_{i=1}^{n}\gamma_i\Xi_i z_i^2 + 2\sum_{i=1}^{n}\Xi_i\left[\alpha_i - \bar{\alpha}_i(X_{[i]}) + \bar{\alpha}_i(0)\sqrt{\frac{\Xi_i(0)}{\Xi_i(z_{[i-1]})|_{z_{[i-1]}=\vartheta_{i-1}(X_{[i-1]})}}}\right]z_i + c_2, \tag{23}$$

where

$$\bar{c}_1 = 1 - \frac{(n-r+1)\varepsilon_1}{2}, \tag{24}$$

$$\gamma_i = \beta_i - 3\sum_{j=i+1}^{n} j\kappa_j - \frac{1}{\varepsilon_{2i}}, \quad i=1,\ldots,n-1 \quad \text{and} \quad \gamma_n = \beta_n - \frac{1}{\varepsilon_{2n}}, \tag{25}$$

$$\bar{\alpha}_1 = \left\{-\Omega_1 - \frac{\beta_1 z_1}{2} - \tfrac{3}{2}\|\bar{\Phi}_{11}(z_1)\|^2 z_1 - \frac{z_1}{2\Xi_1}\mathrm{tr}\{[2\bar{\bar{H}}(0) + r\bar{\bar{H}}_1(z_1)z_1]P\bar{\bar{H}}_1^{\tau}(z_1)\}\right\}_{z_1=x_1},$$

$$\bar{\alpha}_i = \Bigg\{-\Omega_i - \frac{\beta_i z_i}{2} - \frac{\Xi_{i-1}}{2\Xi_i}z_{i-1} - \frac{z_i}{2\Xi_i}\mathrm{tr}\{[2\bar{\bar{H}}(0) + r\bar{\bar{H}}_i(z_{[i]})z_i]P\bar{\bar{H}}_i^{\tau}(z_{[i]})\}$$
$$- \frac{1}{4\Xi_i}\mathrm{tr}\left[\begin{bmatrix} \dfrac{\partial^2\Xi_i}{\partial z_{[i-1]}^2}z_i & 2\dfrac{\partial\Xi_i}{\partial z_{[i-1]}} \\ 2\left(\dfrac{\partial\Xi_i}{\partial z_{[i-1]}}\right)^{\tau} & 0 \end{bmatrix}\begin{bmatrix}\Phi_1\\ \vdots\\ \Phi_i\end{bmatrix}\begin{bmatrix}\Phi_1\\ \vdots\\ \Phi_i\end{bmatrix}^{\tau}\right]$$
$$- \frac{z_i}{2\Xi_i}\sum_{j=1}^{i-1}\frac{\partial\Xi_i}{\partial z_j}(z_{j+1}+\alpha_j+\Omega_j) - \frac{3z_i}{2}\|\bar{\Phi}_{ii}(z_{[i]})\|^2\Bigg\}_{z_{[i]}=\vartheta_i(X_{[i]})}, \quad i=2,\ldots,r-1,$$

$$\bar{\alpha}_r = \Bigg\{-\Omega_r - \frac{\beta_r z_r}{2} - \frac{\Xi_{r-1}}{2\Xi_r}z_{r-1} - \frac{z_r}{2\Xi_r}\mathrm{tr}\{[2\bar{\bar{H}}(0) + r\bar{\bar{H}}_r(z_{[r]})z_r]P\bar{\bar{H}}_r^{\tau}(z_{[r]})\} - \frac{z_r}{4\Xi_r\varepsilon_1}M_r^2$$
$$-\frac{1}{4\Xi_r}\mathrm{tr}\left[\begin{bmatrix} \dfrac{\partial^2\Xi_r}{\partial z_{[r-1]}^2}z_r & 2\dfrac{\partial\Xi_r}{\partial z_{[r-1]}} \\ 2\left(\dfrac{\partial\Xi_r}{\partial z_{[r-1]}}\right)^{\tau} & 0 \end{bmatrix}\begin{bmatrix}\Phi_1\\ \vdots\\ \Phi_r\end{bmatrix}\begin{bmatrix}\Phi_1\\ \vdots\\ \Phi_r\end{bmatrix}^{\tau}\right]$$
$$- \frac{z_r}{2\Xi_r}\sum_{j=1}^{r-1}\frac{\partial\Xi_r}{\partial z_j}(z_{j+1}+\alpha_j+\Omega_j) - \frac{3z_r}{2}\|\bar{\Phi}_{rr}(z_{[r]})\|^2\Bigg\}_{z_{[r]}=\vartheta_r(X_{[r]})},$$

$$\bar{\alpha}_i = \Bigg\{ -\Omega_i - \frac{\beta_i z_i}{2} - \frac{\Xi_{i-1}}{2\Xi_i} z_{i-1} - \frac{z_i}{4\Xi_i \varepsilon_1} M_i^2 - \frac{3z_i}{2} \|\bar{\Phi}_{ii}(z_{[i]})\|^2$$

$$- \frac{1}{4\Xi_i} \mathrm{tr} \left[\begin{bmatrix} \dfrac{\partial^2 \Xi_i}{\partial z_{[i-1]}^2} z_i & 2\dfrac{\partial \Xi_i}{\partial z_{[i-1]}} \\ 2\left(\dfrac{\partial \Xi_i}{\partial z_{[i-1]}}\right)^\tau & 0 \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \vdots \\ \Phi_i \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \vdots \\ \Phi_i \end{bmatrix}^\tau \right]$$

$$- \frac{z_i}{2\Xi_i} \sum_{j=1}^{i-1} \frac{\partial \Xi_i}{\partial z_j} (z_{j+1} + \alpha_j + \Omega_j) \Bigg\}_{z_{[i]} = \vartheta_i(X_{[i]})}, \quad i = r+1, \ldots, n,$$

$$c_2 = \mathrm{tr}\{\bar{\bar{H}}(0) P \bar{\bar{H}}^\tau(0)\} + \sum_{i=1}^n \kappa_i (3 + \varepsilon_{2i} \bar{\alpha}_i^2(0)). \tag{26}$$

Here $\beta_i\,(i = 1, \ldots, n)$ and $\varepsilon_{2i}\,(i = 1, \ldots, n)$ are positive constants to be specified.

In (23)–(26) we have used the following inequality:

$$2 \sum_{i=1}^n \Xi_i \bar{\alpha}_i(0) z_i \sqrt{\frac{\Xi_i(0)}{\Xi_i(z_{[i-1]})}} \leqslant \sum_{i=1}^n \left[\frac{\Xi_i}{\varepsilon_{2i}} z_i^2 + \Xi_i(0) \varepsilon_{2i} (\bar{\alpha}_i(0))^2 \right]$$

$$\leqslant \sum_{i=1}^n \left[\frac{\Xi_i}{\varepsilon_{2i}} z_i^2 + \kappa_i \varepsilon_{2i} (\bar{\alpha}_i(0))^2 \right].$$

If we take α_1, …, α_n as

$$\alpha_i(X_{[i]}) = \bar{\alpha}_i(X_{[i]}) - \bar{\alpha}_i(0) \sqrt{\frac{\Xi_i(0)}{\Xi_i(z_{[i-1]})|_{z_{[i-1]} = \vartheta_{i-1}(X_{[i-1]})}}} \tag{27}$$

and set

$$u = \alpha_n, \tag{28}$$

then from (23) it follows that

$$\mathscr{L}V \leqslant -\bar{c}_1 \|\tilde{x}_{[r+1,n]}\|^2 - \sum_{i=1}^n \gamma_i \Xi_i z_i^2 + c_2. \tag{29}$$

From the above design procedure we see that the key point is how to choose the positive parameters ε_1, κ_i, ε_{2i} and β_i ($i = 1$, …, n) such that

$$\bar{c}_1 > 0, \quad c_2 \geqslant 0, \quad \gamma_1 > 0, \ldots, \gamma_n > 0. \tag{30}$$

The following lemma tells us a method specifying these parameters.

Lemma 1. *For any given positive constants κ_i and ε_{2i} $(i=1,\ldots,n)$, if $0 < \varepsilon_1 < 2/(n-r+1)$, $\beta_n > 1/\varepsilon_{2n}$ and $\beta_i > 3\sum_{j=i+1}^n j\kappa_j + 1/\varepsilon_{2i}$ $(i = 1, \ldots, n-1)$, then the inequalities in (30) hold.*

Proof. The result comes directly from (24) to (26) and conditions $0 < \varepsilon_1 < 2/(n-r+1)$, $\kappa_i > 0$ $(i=1,\ldots,n)$, $\varepsilon_{2i} > 0$ $(i = 1, \ldots, n)$, $\beta_n > 1/\varepsilon_{2n}$ and $\beta_i > 3\sum_{j=i+1}^n j\kappa_j + 1/\varepsilon_{2i}$ $(i = 1, \ldots, n-1)$. $\square$

We now summarize the main result of this paper in the following theorem.

Theorem 2. *Consider the nonlinear stochastic system* (4). *Suppose Assumptions* A.1–A.3 *hold. Then a reduced-order observer-based feedback controller can be constructively designed so that the closed-loop system admits an almost surely P unique solution on* $[0, \infty)$, *and is bounded in probability. Besides, when* $\varphi(0) = 0$, *the closed-loop system is asymptotically stable in the large.*

Proof. Actually, we can design control by (27) and (28). In this case, we have

$$\mathscr{L}V \leqslant -c_1 V + c_2, \tag{31}$$

where $c_1 = \min(\bar{c}_1 \lambda_{\max}^{-1}(P), \gamma_1, \ldots, \gamma_n)$. Note that V is positive definite, radially unbounded and twice continuously differentiable function in terms of states of the closed-loop system, then by Theorem 1 we conclude that there exists an almost surely P unique solution on $[0, \infty)$ to the closed-loop system, and the solution process is bounded in probability.

If $\varphi(0) = 0$, then we have $\bar{\bar{H}}(0) = 0$, $\bar{\Phi}_i(0) = 0$ and $\bar{\alpha}_i(0) = 0$. This leads to $c_2 = 0$ and

$$\mathscr{L}V \leqslant -c_1 V,$$

which together with Theorem 1 implies that the zero solution of the closed-loop system is asymptotically stable in the large. □

Remark 2. Clearly, from (27) we know that the virtual control laws $\alpha_i (i = 1, \ldots, n-1)$ and actual control law $u = \alpha_n$ preserve the equilibrium point of the open-loop nonlinear system.

Roughly speaking, the smaller the parameters κ_i and ε_{2i} $(i = 1, \ldots, n)$ are, the smaller c_2 is. This together with (31) tells us that if we want to get a small static upper bounded for the states of the closed-loop systems, then we should take small κ_i and ε_{2i} $(i = 1, \ldots, n)$.

5. An example

We now give an example to illustrate the design method proposed above.

Consider the following two-dimension stochastic nonlinear system:

$$\mathrm{d}x_1 = x_2\,\mathrm{d}t + x_1^2\,\mathrm{d}t + x_1\,\mathrm{d}w,$$

$$\mathrm{d}x_2 = u\,\mathrm{d}t + x_1^2\,\mathrm{d}t + x_1\,\mathrm{d}w, \qquad y = x_1,$$

where w is a scalar Brownian motion satisfying Assumption A1.

In this case, $n = 2$, $r = 1$, $A_1 = 0$, $B_1 = 1$, $C_1 = 1$, $D = A_1 - GB_1C_1^{\tau} = -g$, and the estimate of x_2 is given by $\hat{x}_2 \triangleq \xi + gx_1$, where ξ is generated by

$$\dot{\xi} = -g\xi + u + x_1^2 - gx_1^2 - g^2 x_1.$$

Here $G = g$ is a design parameter.

Following the design method described above, take

$$z_1 = x_1, \qquad \Xi_1 = \kappa_1,$$

$$\alpha_1(x_1) = -x_1^2 - \frac{\beta_1 x_1}{2} - \frac{3x_1}{2} - \frac{x_1}{\Xi_1}\frac{(1-g)^2}{2g}x_1^3 = -x_1^2 - \frac{(3+\beta_1)x_1}{2} - \frac{(1-g)^2}{2g\kappa_1}x_1^4,$$

$$z_2 = \xi - \alpha_1(x_1),$$

$$\Xi_2(z_1) = \frac{\kappa_2}{1 + \Xi_1^{-1}|1 - \partial\alpha_1/\partial x_1|^2} = \frac{\kappa_2}{1 + \kappa_1^{-1}|1 + 2z_1 + (3+\beta_1)/2 + 2(1-g)^2/(g\kappa_1)z_1^3|^2},$$

$$\alpha_2(x_1,\xi) = \left\{ -x_1^2 - g\xi - gx_1^2 - g^2x_1 - \frac{\partial\alpha_1}{\partial x_1}(x_1^2 + \xi + gx_1) - \frac{\beta_2 z_2}{2} - \frac{\Xi_1}{\Xi_2}x_1 - \frac{z_2}{2\Xi_2}\frac{\partial \Xi_2}{\partial z_1}(z_2 + \alpha_1 + \Omega_1) \right.$$

$$\left. - \frac{1}{2}\frac{\partial^2\alpha_1}{\partial x_1^2}x_1^2 - \frac{1}{4\Xi_2}\operatorname{tr}\left[\begin{bmatrix} \dfrac{\partial^2\Xi_2}{\partial z_1}^2 z_2 & 2\dfrac{\partial\Xi_2}{\partial z_1} \\ 2\dfrac{\partial\Xi_2}{\partial z_1} & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_1 - \dfrac{\partial\alpha_1}{\partial x_1}x_1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_1 - \dfrac{\partial\alpha_1}{\partial x_1}x_1 \end{bmatrix}^\tau\right] \right\}_{z_1=x_1,\, z_2=\xi-\alpha_1(x_1)},$$

where β_1, β_2, κ_1 and κ_2 are design parameters. Then by Theorem 2, under control $u=\alpha_2(x_1,\xi)$ the closed-loop system is asymptotically stable in the large, provided that the design parameters are chosen such that $g>0$, $\kappa_1>0$, $\kappa_2>0$, $\beta_1>6\kappa_2$, $\beta_2>0$.

6. Concluding remarks

In this paper, we developed a new reduced-order observer-based backstepping design procedure, and presented an output-feedback stabilization controller for strict-feedback stochastic nonlinear systems. The controller designed guarantees the closed-loop system asymptotically stable in the large when the nonlinearities and stochastic disturbance equal zero at the equilibrium point of the open-loop system, and bounded in probability, otherwise. Besides, the controller preserves the equilibrium point of the nonlinear system.

References

[1] H. Deng, M. Krstić, Output-feedback stochastic nonlinear stabilization, IEEE Trans. Automat. Control 44 (1999) 328–333.

[2] H. Deng, M. Krstić, R. Williams, Stabilization of stochastic nonlinear systems driven by noise of unknown covariance, IEEE Trans. Automat. Control 46 (2001) 1237–1253.

[3] P. Florchinger, Lyapunov-like techniques for stochastic stability, SIAM J. Control Optim. 33 (1995) 1151–1169.

[4] R.A. Gihman, A.V. Skorohod, Stochastic Differential Equations, Springer, New York, 1972.

[5] Z.P. Jiang, Global output feedback control with disturbance attenuation for minimum-phase nonlinear systems, Systems Control Lett. 39 (2000) 155–164.

[6] R.Z. Khas'minski, Stochastic Stability of Differential Equations, S and N International Publishers, Rockville, MD, 1980.

[7] H.J. Kushner, Stochastic Stability and Control, Academic Press, New York, 1967.

[8] Y.G. Liu, Z.G. Pan, S.J. Shi, Output feedback control design for strict feedback stochastic nonlinear systems under a risk-sensitive cost criterion, Proceedings of the 40th IEEE Conference on Decision and Control, 2001, pp. 1269–1274.

[9] Z.G. Pan, T. Basar, Backstepping controller design for nonlinear stochastic systems under a risk-sensitive cost criterion, SIAM J. Control Optim. 37 (1999) 957–995.

[10] Z.G. Pan, Y.G. Liu, S.J. Shi, Output feedback stabilization for stochastic nonlinear systems in observer canonical form with stable zero-dynamics, Sci. China (Ser. F) 44 (2001) 292–308.

[11] E.P. Wang, H.S. Qin, S.L. Wang, Introduction to Linear Control System Theory, Guangdong Science and Technology Press, Guangdong, 1991.